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RELATIONS BETWEEN STATURE AND CHEST-GIRTH FORMULA OF NORMALITY AND NORMAL VALUES

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Recently we have graphically and analytically demonstrated[1] that linear relation runs between height and weight. It seemed useful to find out whether the same relation runs between height and chest-girth, using the method described in quoted paper.[1]

The available material for such a study consisted of data on stature and chest-girth given in Livi's "Antropometria Militare," Rome, 1896.[2] The data used are given of age, in detail, in table 1, which shows measurements made on males from 20 years and 2 months to 21 years and 2 months. The 255,181 used as subjects were Italian recruits, the most of them countrymen born within the quinquennium 1859–1863.

Here is how we have proceeded.

Spaced horizontally, from the origin O, along the abscissa OX, are placed the successive averages of chest-girth from 85 cm. to 100 cm., inclusive. Spaced vertically from the same origin O, along the ordinate OY are placed the corresponding statures (1 cm. = 1 cm. of the stature).

The portion of plotted curves corresponding to the largest observations (75 per cent of total number), and labelled AB shows a straight line, which forms with the abscissa OX an angle of 76° 30′, and cuts the negative part of the abscissa at 0.4 cm. from the origin.

Now we consider the straight line algebraically. If, in the general equation of a right line $Y-Y_0 = m(X-X_0)$, we put

$Y = S$ = measured stature in cm. from O,
$Y_0 = S_0$ = conventional initial stature = 154 cm.,
$X = Ch.G$ = chest-girth corresponding to measured stature,
$X_0 = Ch_0.G_0.$ = conventional initial = 85 cm.,

we have this following equation:

$$S-S_0 = m[X-(X_0-0.4 \text{ cm.})] \qquad (1)$$

m value of the trigonometric tangent of the angle which the right line

AB forms with the abscissa OX (angle 76° 30′ = 4.165) 0.4 cm. distance from the origin.

Referring to equation (1) we have

$$S-S_0 = 4.165\ (X-X_0 + 0.4\ \text{cm.})$$

from which, for X value of researched chest-girth

$$S-S_0 = 4.165\ X-354.025 + 1.666$$

whence

$$X = \frac{S-S_0 + 354.025 + 1.666}{4.165}. \qquad (2)$$

Letting now (Table 1) S = the stature; *Ch.G.*, *Ch.'G.'* = the experimental and theoretical values corresponding to each centimeter of stature; D' = the difference of two successive theoretical values of chest-girth; D = the difference between experimental and theoretical values we see that the chest-girth succeeds each other in an arithmetical progression whose rate is 0.24 cm. Hence, it is argued, relation between stature and chest-girth is linear.* One can, therefore, derive the following law:

*"In young Italians of 20–21 years starting from stature of 154 cm. and corresponding chest-girth of 84.6 cm. chest girth increases in arithmetical progression at the rate of 0.24 cm.** for every centimeter increase of stature,"* or expressed in symbols

$$Ch.G. = 84.6 + (0.24\delta) \qquad (3)$$

where δ equals the number of centimeters by which the considered stature exceeds the initial stature of 154 cm.

Formula (3) can be deduced directly from formula (1).

If in the equation

$$S-S_0 = m[X-(X_0-0.4\ \text{cm.})] \qquad (I)$$

we put $S-S_1 = \delta$; X = chest-girth, we have

$$\delta = 4.165\ (Ch.G.-84.6)$$

hence

$$Ch.G. = 84.6 + \frac{1\delta}{4.165}, \qquad Ch.G. = 84.6 + 0.24\delta.$$

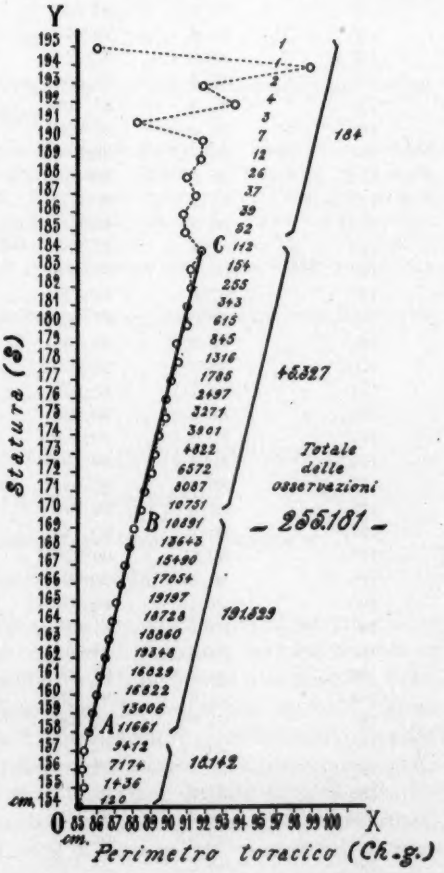
Let us now verify the exactness of the new formula (3) in relation with the first one (1) taking, at random, between 158 and 169 cm. the successive

* This relation can be deduced from the little diagram drawn by Livi (Loc. cit., Tab. IV, Fig. 14) with five-centimeter class-intervals for the statures and one-cm. classes for chest-girth. Costantino Bresciani, on the contrary, spoke about this relation by calculating the coefficient of correlation (r = 0.34) and the two respective equations of regression for chest-girth and stature. (See, *Giornale degli Economisti*, **38** (ser. 2), Roma, 1909.)

** The rate (0.24), stated above, corresponds to that obtained by Bresciani proceeding by different methods. He wrote, "The equations of regressions tell us that increasing the stature of one centimeter the relative median values of chest-girth grows about a quarter of centimeter.

statures, for examples, 161–162–163 cm.; chest-girth obtained by formula (3) are, respectively, 86.28–86.52–87.76 cm., viz., same values tabulated in table 1 which differ from corresponding empirical values by an error not greater than a millimeter.

Consequently, then, we can conclude that the formula (3) must be considered the formula of normality of the relation between stature and chest-girth. But this formula is exact only for Italians 20 years old having statures comprehended between 156 and 159 cm. corresponding to 75 per cent of total series in consideration. For the statures above and below, the corresponding theoretical means of chest-girth differ from the means of experimental data on an error which is the greater as the statures under consideration are far from the statures 156 to 169. In effect for the statures comprehended between 169 and 180 cm. (cf. Table 1, Column 3) the difference is about two millimeters, while for the statures above to 180 cm. this difference may exceptionally surpass five centimeters, being normally below two cm. It will be observed that such differences are particularly due to the fewer observations (0.07 per cent) corresponding to taller statures. Yet our "formula of normality" results sufficiently accurate for 99.93 per cent of cases.



The case is not here to consider the relation between stature and increasing values (centimeter by centimeter) of chest-girth. For anthropometrical researches this relation is less important than the one we have considered.

Summary.—In this paper the author undertakes to assay the relation

TABLE 1

TABLE OF RELATIVE CHEST-GIRTH (Ch.'G.') FOR INDIVIDUAL STATURES (S) MEASURED CENTIMETER BY CENTIMETER FROM 154 CM. TO 195 CM., CALCULATED ACCORDING TO THE FORMULA OF THE NORMALITY (SEE TEXT)

VALUES OF STATURE S (IN CM.)	MEDIUM VALUES OF CHEST-GIRTH CORRESPONDING TO EACH CM. OF STATURE EXPERIMENTAL CH.G. (IN CM.)	CALCULATIONS CH.'G.' (IN CM.)	D = CH.G. − CH.'G.'	D' = CH.'G.' − CH."G."
154	85.0	84.599	0.410	+0.24
155	85.3	84.839	0.539	
156	85.2	85.079	0.121	
157	85.3	85.319	0.019	
158	85.6	85.559	0.041	
159	85.8	85.799	0.001	
160	86.0	86.039	0.039	
161	86.2	86.279	0.079	
162	86.5	86.519	0.019	
163	86.7	86.759	0.059	
164	86.9	86.999	0.099	
165	87.1	87.239	0.139	
166	87.4	87.479	0.079	
167	87.6	87.719	0.119	
168	87.9	87.959	0.059	
169	88.1	88.199	0.099	
170	88.5	88.439	0.061	
171	88.7	88.679	0.021	
172	89.1	88.919	0.181	
173	89.3	89.159	0.141	
174	89.5	89.399	0.101	
175	89.9	89.639	0.241	
176	89.9	89.879	0.021	
177	90.2	90.119	0.081	
178	90.6	90.359	0.241	
179	90.5	90.599	0.099	
180	91.1	90.839	0.261	
181	91.1	91.079	0.021	
182	91.3	91.319	0.019	
183	91.3	91.559	0.259	
184	91.9	91.799	0.101	
185	91.0	92.039	1.039	
186	90.9	92.279	1.379	
187	91.6	92.519	0.919	
188	91.1	92.759	1.659	
189	91.9	92.999	1.099	
190	92.0	93.239	1.239	
191	88.3	93.479	5.179	
192	93.8	93.719	0.081	
193	92.0	93.959	1.959	
194	98.0	93.999	4.001	
195	96.0	94.239	1.761	

which exists between stature and chest-girth. He graphically and analytically demonstrates that linear relation runs between these two characters, like between height and weight. By using individual measurements of 255,181 subjects, the author can derive the following law: "In young Italians of 20–21 years starting from stature of 154 cm.; and corresponding chest-girth of 84.6 cm., chest-girth increases in arithmetical progression at the rate of 0.24 cm. for every centimeter increase of stature." Or, expressed in symbols

$$Ch.G. = 84.6 + (0.24 \times \delta)$$

where δ equals the number of centimeters by which the considered stature exceeds the initial stature of 154 cm.

[1] Frassetto, F. *Height-Weight Index of Build or Robustness. Eugenics in Race and State.* Vol. ii, p. 36—*The Second International Congress of Eugenics.* Baltimore, Williams & Wilkins Company, 1923. For further information on this subject, may consult the following papers by the author: "Delle relazioni fra il peso e la statura nell 'uomo adulto," *Atti R. Acc. Naz. Lincei. Cl. Sci.*, **30** (s. 5a), fasc. 12, Roma, 1921—"Altre forme della legge che vincola i pesi alle stature negli adulti," *Ibid.*, **31** (ser. 5a), fasc. 8, Roma, 1922.

[2] Livi, R. *Antropometria Militare*, Roma, Presso il Giornale Medico del R. Esercito, 1898.

SOCIAL SELECTION IN A MIXED POPULATION

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In the development of human types, the effect of social selection is an important factor which has often been disregarded, perhaps because of difficulty in measuring the qualities which determine the selection practised, or, indeed, of discovering what these qualities may be. Thus, European attempts to segregate it have ordinarily given indecisive results.[1] In the present study, which is concerned with mixed Negroes in this country,[2] it has become apparent that there is a definite type of selection practised by American Negroes, based largely on skin color, which manifests itself in many aspects of the life of these people,[3] particularly in marriage.

The first indication of this selection I had was while measuring students at Howard University, where each individual was asked to state whether his father or his mother was the lighter. The results from this inquiry were:

Father lighter	115	30.3%
About the same color	50	13.2%
Mother lighter	215	56.5%

This seemed to indicate a strong trend toward dark men mating with light women, a tendency which, if continued, must in time affect the physical type of the American Negro population.

During the past winter, there have been measured about 400 Negro families, parents and children, in the Harlem district of New York City. Of these, 174 married couples were available for a preliminary quantitative study of the manner in which this color selection operates in marriage. Skin color is taken with a color-top[4] on which there are disks of black, red, yellow and white. The top, when adjusted and spun, gives a mixture of these which matches the skin of the individual being measured. Pigmentation was taken on the upper outer arm, and a white sheet of paper with a hole in it was placed over the arm so that judgment as to the color would not be affected by the tints surrounding the spot matched, while the top was spun on another sheet of white paper. The red in the top is not pure, but when compared with Ridgeway[5] gives red, 41%, black, 59%. The necessary corrections have been made in all the figures given below.

If we first consider the proportions of the comparative color of husband and wife as observed, it is seen that these check to a remarkable extent with the statements of the Howard University students, the proportions for the Harlem population being:

Father lighter	49	29.0%
Same color	25	14.5%
Mother lighter	100	56.5%

This seems to show that the tendency for light men to marry dark women observed through the statements of the Howard students is valid, but it is of interest to see the extent to which further analysis throws light on the way in which this tendency operates. If we consider the average values for the black element[6] of the color top (in terms of percentages of the total surface of the top) and their variabilities, these are:

Husbands	72.5%	±13.33
Wives	67.7%	±13.56

and it is thus seen that the men who are party to the matings are darker than the women, while the variabilities for the two sexes are practically identical. An examination of the percentage of black in the top for a series of 64 unmarried Negro women between 15 and 23 years of age

and for another series of 40 Negro men between 16 and 22 years of age from the same population, gives average values as follows:

Men	76.3%
Women	76.4%

This seems to show that there is no sex difference between unselected series of males and females with respect to this trait, and that the lighter average color of the wives of my series, when compared to that of their husbands, is actually the result of this factor of social selection.

FIG. I -- AVERAGE SKIN COLOR OF WIVES (% BLACK) FOR INCREASINGLY DARK HUSBANDS
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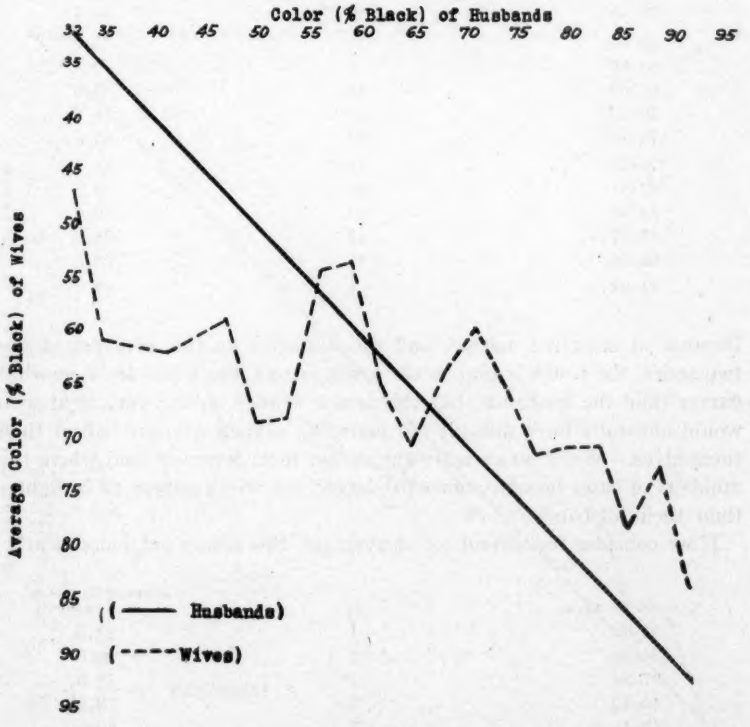
Color (% Black) of Husbands
32 35 40 45 50 55 60 65 70 75 80 85 90 95
Average Color (% Black) of Wives
35 40 45 50 55 60 65 70 75 80 85 90 95
(—— Husbands)
(---- Wives)

Considering in detail the color of each husband with respect to that of his wife, we have the results given in table 1. The black element tends to mass toward the upper ranges, as may be observed, and, therefore, the coefficient of correlation was not calculated, it being considered advisable

to give the results rather in terms of the average color (expressed in amounts of percentage of black) of the wives for each color husband, and vice versa. For the first, the results are as follows (given graphically in Fig. I):

% BLACK OF HUSBANDS	NO.	AVERAGE % BLACK OF WIVES
31–33	1	47.0
34–36	3	61.0
37–39	.	..
40–42	3	62.0
43–45	.	..
46–48	1	59.0
49–51	7	68.4
52–54	2	68.0
55–57	7	54.3
58–60	8	53.4
61–63	10	65.0
64–66	5	70.4
67–69	15	63.8
70–72	10	59.9
73–75	15	65.0
76–78	13	71.2
79–81	22	70.6
82–84	21	70.6
85–87	17	78.0
88–90	10	72.5
91–93	4	83.7

Because of assortive mating, and the difference in the averages of the two series, the result is that in the lower ranges, the wives are somewhat darker than the husbands; but this is not strange, as the very light men would naturally have difficulty in marrying women who are lighter than themselves. When we come to the darker men, however, and where the numbers of cases become somewhat larger, the wives appear to be lighter than their husbands.

If we consider the second set of averages, this comes out more clearly:

% BLACK OF WIVES	NO.	AVERAGE % BLACK OF HUSBANDS
31–33	1	83.0
34–36	1	59.0
37–39	1	71.0
40–42	3	78.0
43–45	3	69.0
46–48	8	60.1
49–51	6	69.5
52–54	11	65.8
55–57	7	65.9
58–60	18	68.3

61–63	12	69.7
64–66	9	71.3
67–69	7	65.9
70–72	11	74.4
73–75	16	69.5
76–78	14	77.0
79–81	15	79.0
82–84	13	74.9
85–87	10	85.1
88–90	6	83.0
91–93	2	92.0

FIG. II -- AVERAGE SKIN COLOR OF HUSBANDS (% BLACK) FOR INCREASINGLY DARK WIVES

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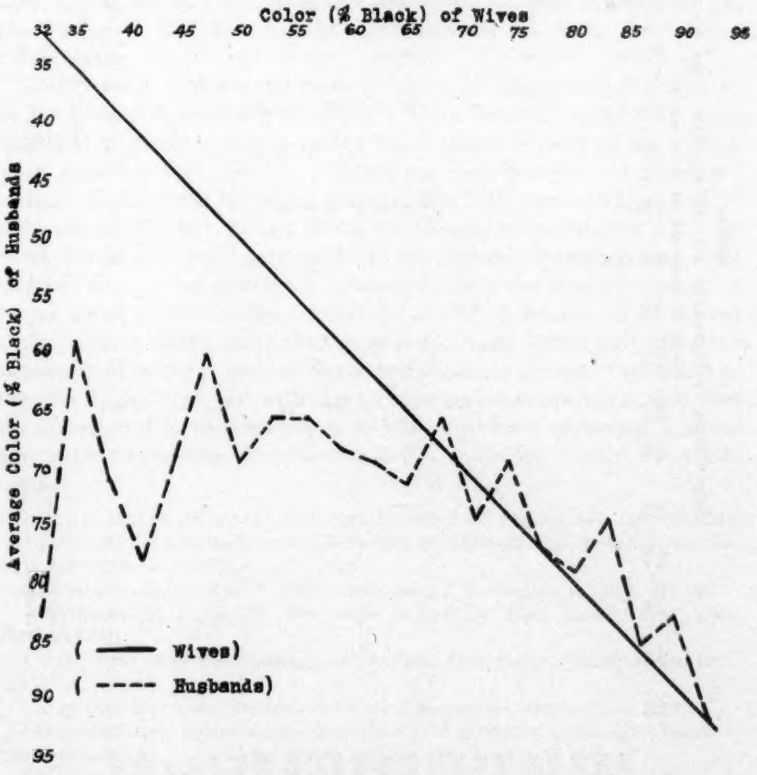
Color (% Black) of Wives
32 35 40 45 50 55 60 65 70 75 80 85 90 95
Average Color (% Black) of Husbands
35 40 45 50 55 60 65 70 75 80 85 90 95
(—— Wives)
(---- Husbands)

Here, in thirteen of the classes, the husbands are darker than their wives; in two (including the darkest) they are the same color; while in only six

TABLE 1

SKIN COLOR OF HUSBANDS AND WIVES, NEW YORK NEGROES

COLOR OF WIVES (IN % OF BLACK)	COLOR OF HUSBANDS (IN % OF BLACK)																				
	31–33	34–36	37–39	40–42	43–45	46–48	49–51	52–54	55–57	58–60	61–63	64–66	67–69	70–72	73–75	76–78	79–81	82–84	85–87	88–90	91–93	TOTAL
31–33																		1				1
34–36										1												1
37–39														1								1
40–42														1	1					1		3
43–45									1					1			1					3
46–48	1	1		1						1	1							2	1			8
49–51									1	1			1		1	1		1				6
52–54							1		2	3		1			1	1	1			1		11
55–57				1					1		1				3		1					7
58–60						1	1		1	1	1		6	1	1	1	2	1	1			18
61–63		1									2		2	2	1		2	2				12
64–66								1	1		1	1	1			1	1			1	1	9
67–69							2						2	1	1		1					7
70–72								1			1			1	1	2	1	1	2	1		11
73–75		1					2				2	1	1	1	1	2	1	1	3			16
76–78										1		1	2			1	5	2	1	1		14
79–81											1			1	2	2	3	4		2		15
82–84				1			1					1			1	1	2	4	1	1		13
85–87															1			2	5	1	1	10
88–90																1	1		3	1		6
91–93																					2	2
Totals	1	3	.	3	.	1	7	2	7	8	10	5	15	10	15	13	22	21	17	10	4	174

	MEAN	STANDARD DEVIATION
Husbands	72.5	±13.11
Wives	67.7	±13.56

are they lighter. The relationship between the two series also comes out somewhat more clearly if the graph of figure II be consulted, where it can be seen to be strongly curvilinear. The darkest women, again, as might be expected, seem to marry men of about their color or lighter, but the lighter women, and those not pronouncedly dark, marry men darker than themselves.

An attempt was also made to see whether the selection among the Negroes is on the basis of color only, or is also made with regard to the extent to which other traits are Negroid or non-Negroid. Lip thickness and the width of the alae of husbands and wives were correlated (there being no skewness to the curves for these traits), but the resulting correlations show practically nothing, being, for lip thickness, +0.22, and for nose width, +0.13. The means for the wives are well below those for the husbands, but this can be accounted for as being due to sex differences.

Thus it seems that a social selection among the Negroes in this country on the basis of skin color is occurring. What the effect on future generations will be depends largely on the continuation in force of the present *mores* regarding such selection. If they continue, however, the daughters of these matings will be, in the main, darker than their mothers, and, if they are selected in turn and choose for themselves still darker men, the effect will be that the American Negro population will become nearer the Negroid type as far as color is concerned, since the relative amount of Negro blood in the Negro population will be increased. Genealogical studies I have made[7] show that there is too much White and American Indian blood in the American Negro population to permit of reversion to the pure Negro type, but with this social selection in operation, and with the stoppage of further crossing with Whites which is occurring, it seems reasonable to conclude that there will not be change toward the White norm.

[1] As, e.g., that by Pearson, K., *Proc. Royal Society "A,"* London, **187**, 1896, 273–275.

[2] This study has been made under a Fellowship of the Board of the Biological Sciences, National Research Council.

[3] Herskovits, M. J., *Am. J. Phys. Anthropology, Washington*, **9**, 1926, 321–327.

Herskovits, M. J., in *The New Negro* (edited by Alain Locke), New York, **1926**, 353–360.

[4] Todd, T. W., and Van Gorder, L. A., *J. Phys. Anthropology, Washington*, **4**, 1921, 238–260.

[5] Ridgeway, R., *Color Standards and Color Nomenclature*, Washington, **1912**.

[6] Only black is considered in this paper, since it, of all the four colors, shows comparative darkness best. The higher values indicate the darker skin colors.

[7] Herskovits, M. J., *Pub. A. Sociological Soc., Chicago*, **1926**, 77–82.

A PRELIMINARY REPORT ON THE MAGNETIC SUSCEPTIBILITIES OF SOME GASES

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Accurate information about the magnetic properties of matter is greatly needed. The susceptibilities of most substances are known very imperfectly, if at all, and the information we have does not agree with any simple theory. The lack of precise data is particularly noticeable when we consider the simple gases and vapors, yet it is just from the knowledge of these that we hope to advance in our theories about magnetism.

At the suggestion of Professor R. A. Millikan, the writer set out to determine as accurately as possible the magnetic susceptibilities of the simple gases. The method used is briefly as follows: If a rod-shaped body is suspended in an inhomogeneous magnetic field it experiences a moment of force expressed by the equation:

$$L = (k_o - k_s) \int_v (1/2 \text{ grad } H^2 \times \bar{r}) dv$$

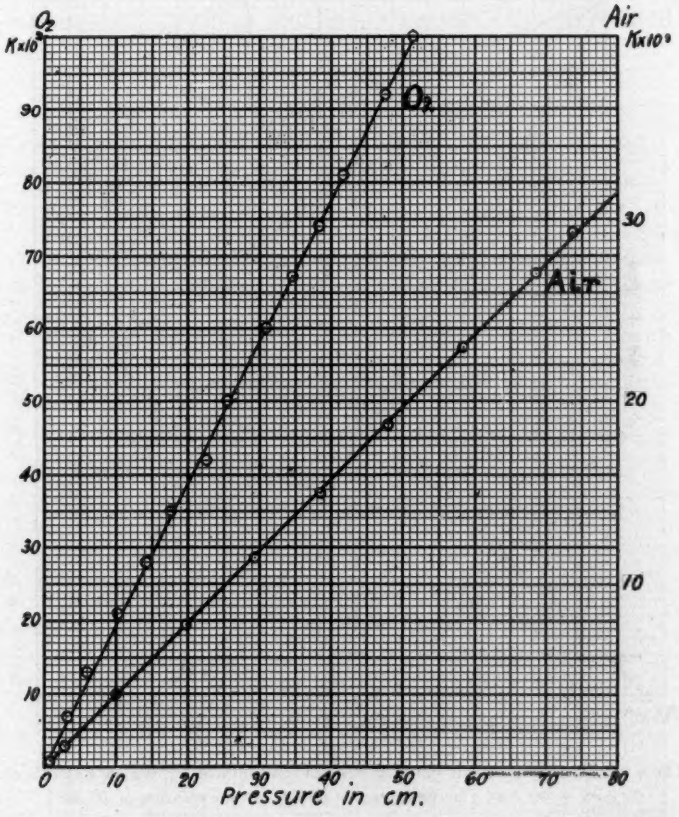
where k_o is the volume susceptibility of the test body, k_s the volume susceptibility of the surroundings, H the strength of the magnetic field, and $\bar{r}$ the radius vector from the axis of rotation to the element of volume, dv. The integration is performed over the volume of the test body. By maintaining the field constant and the position of the test body fixed, the moment of force is directly proportional to $(k_o - k_s)$, and is, if taken up by twisting the suspension (in our case a quartz fiber), proportional to the torsion of the fiber. As can be seen, this is inherently a differential method. In its simplest form it was used by Faraday and has since been used by many others. It is capable of great sensitivity and fine accuracy. One of its drawbacks, the sensitivity to thermal changes, has been greatly reduced by using a diamagnetic test body.

With such an apparatus we proceeded to determine the volume susceptibility of some of the common gases at room temperature. Samples of results for air, O_2, N_2, CO_2, and H_2, are shown on figures 1 and 2.

The absolute value in each case depends on the volume susceptibility of oxygen, which, according to Wills and Hector[1] is 1.447×10^{-7} at a temperature of 20° Centigrade and a pressure of 76 cm. of mercury. From the graphs it can be seen that the volume susceptibility of a gas is directly proportional to its pressure, hence proportional to the number of molecules present. This indicates that the magnetic properties of these gases are strictly molecular in the old sense.

Although this result is one which would be generally expected, up to

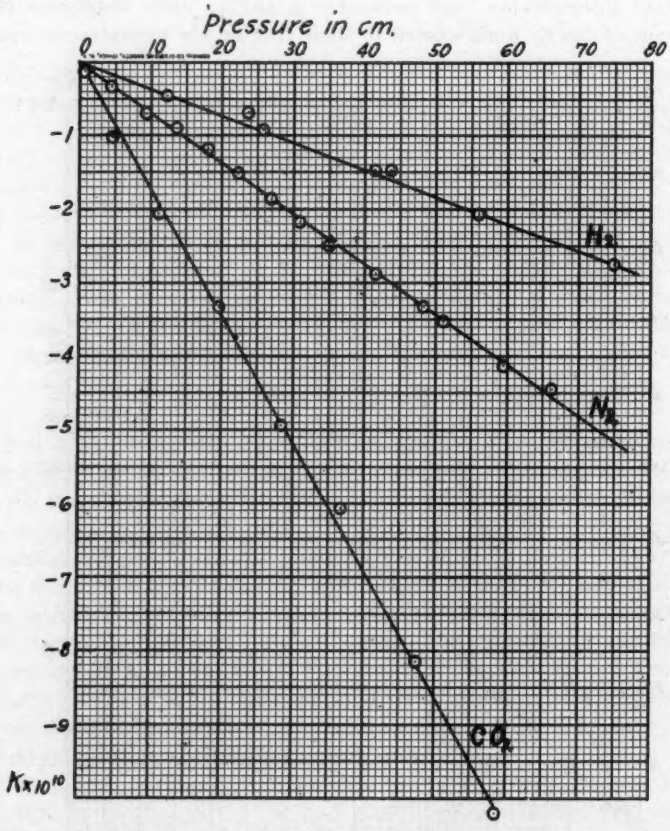
the present time there has been no work to clearly demonstrate it. Indeed, the work of Glaser,[2] while showing a linear progression of susceptibility with density for high densities, shows definitely a failure of that law at low densities, where it should have most unequivocally come into evidence; for as the density diminishes the intermolecular properties should sink into insignificance, and molecular properties alone determine the behavior. Glaser's work seemed to show that at low pressures suscepti-



bilities are out of line with this requirement, a fact for which he found an explanation in the assumption of spacial quantization. The present work, however, shows unambiguously that from the lowest pressures up to atmospheric susceptibility is, in fact, proportional to density.

A second contribution presented in the work is the precision with which the magnetic properties of the gas molecules have been determined. For

a given density the precision appears to be at least twice as great as heretofore attained. The need for greater reliability in the determination of the susceptibility of gases is shown by the fact that two of the latest and most careful determinations[1,3] of the susceptibility of O_2 differ by $4^1/_2$ per cent, although each observer claims an accuracy of $^1/_{10}$ per cent.



These values depend on the value of the susceptibility of air, which was taken as 2.98×10^{-8} at a temperature of 24.5°C. and a pressure of 76 cm.

After the work of the present report had been completed a note by Lehrer[4] appeared which reports results in agreement with mine and in disagreement with Glaser's.

[1] Wills and Hector, *Phys. Rev.*, **23**, p. 209, 1924.
[2] A. Glaser, *Annalen der Physik.*, **75**, p. 459, 1924.
[3] Také Soné, *Phil. Mag.*, **39**, p. 305, 1920.
[4] E. Lehrer, *Zeischrift fur Physik*, **73**, p. 155, 1926.

A POSSIBLE EXPLANATION OF THE "GLASER EFFECT"

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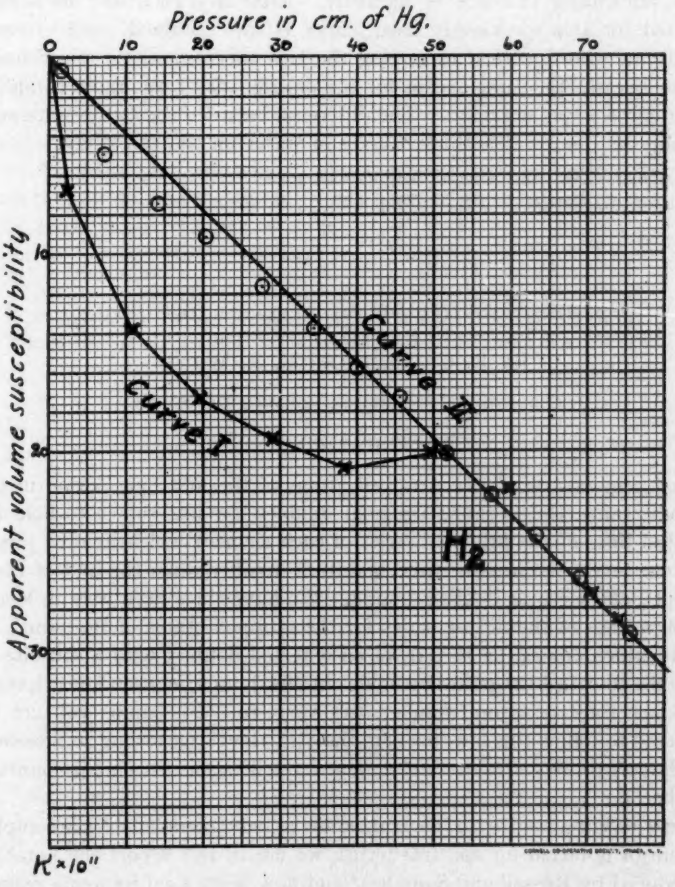
In a previous report[1] I have shown that the specific susceptibilities of H_2, N_2 and CO_2 are independent of pressure over a range from zero to one atmosphere. The fact that this contradicts results obtained by



Glaser[2] caused me to look for a possible source of error sufficient to produce the deviation reported by him, and, by chance, one was discovered which

gives a perfect "Glaser effect." Since both Glaser and I use the same method, viz., measure the moment of force exerted on a test body in an inhomogeneous magnetic field, the main sources of error should be the same for both.

After several runs with H_2, which gave constant specific susceptibility, I left the apparatus for a few days, and on returning obtained a good "Glaser effect," see curve I. Two more runs gave similar results. On examination of the cleaning train, which furnishes dry gas for my apparatus, the P_2O_5 in the last drying tube was found to have acquired a slight glaze, indicating presence of moisture. After dry P_2O_5 had been substituted for that previously used, curve II was obtained. All external conditions were identical in getting the two curves, and the purification of the gas was the same except for the drying, which was more complete when curve 2 was obtained. The difference between the two results may possibly be due to an adsorbed layer of water on the test body.

[1] Hammar, G. W., this issue of these PROCEEDINGS.

[2] Glaser, A., *Annalen der Physik*, **75**, p. 459.

STRIPPED ATOMS OF THE FIRST LONG PERIOD

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In a previous report[1] on stripped atoms of the first long period in the periodic table, the so-called irregular doublet law was shown to hold for the first lines of the principal series of each element which involve jumps between levels of the same total quantum number when the first of these stripped-off electrons is being bound. We have since been able to locate the first lines of the diffuse series for the stripped atoms of the same elements which consist as in Ca_{II} of an inverted doublet with a satellite.

As the so-called irregular law holds only for jumps between levels having the same total quantum number, the $4^2P_{1,2}$ to $3^2D_{2,3}$ jumps will not be expected to follow this law, for here the $4^2P_{1,2}$ levels according to Sommerfeld have a total quantum number 4, and the $3^2D_{2,3}$ levels a total quantum number 3.

Since recent advances in spectrum analysis has demanded a more complex system of notation for spectral terms, we use in this report the notation introduced by Russell and Saunders[2] and now being used by many others. In the above terms, the integer prefixed to the term gives the total quantum number of the orbit, the capitalized letter gives the azimuthal quantum

number, i.e., the type of term and the superscript to the left and subscript to the right give the multiplicity of the term and the corresponding Landé inner quantum numbers, respectively.

From Fowler's report, "Series in Line Spectra,"[3] it has long been known that in K_I state Ca_{II} the lowest or normal and of the atom is a single 4^2S_1 level and, therefore, requires the single valence electron to be an "s" electron. In the Moseley diagram, figure 1, it may be seen that the $3^2D_{2,3}$ levels have changed from the third position in K_I to the second position in Ca_{II}, while with the new data on scandium they have shifted to the position of lowest energy level in Sc_{III}.

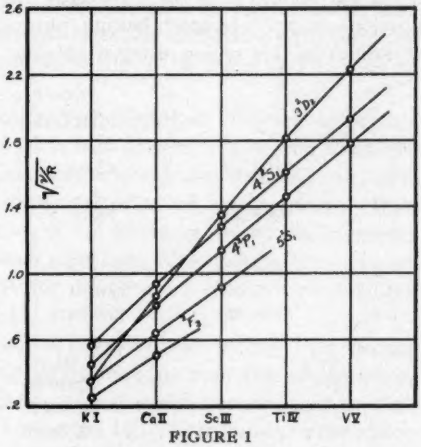

FIGURE 1

It is apparent that here is a strong confirmation that the "d" electron is the most tightly bound electron through this group of elements. This

TABLE 1

SCANDIUM$_{III}$

INT.	λ	ν	Δν	TERMS
2	2734.83	36565.3		$4^2P_2-4^2S_1$
			474.3	
3	2699.81	37039.6		$4^2P_1-4^2S_1$
3	2012.9	49680		$4^2P_2-5^2S_1$
			473	
5	1993.9	50153		$4^2P_1-5^2S_1$
4	1610.3	62100		$3^2D_2-4^2P_1$
5	1603.3	62372	478	$3^2D_3-4^2P_2$
1	1598.0	62578	206	$3^2D_2-4^2P_2$

TERM	VALUE	$\sqrt{\nu/R}$
4^2S_1	178080.2	1.274
4^2P_1	141514.9	1.135
4^2P_2	141040.6	1.134
3^2D_2	203618.6	1.362
3^2D_3	203412.6	1.361
5^2S_1	91362.0	0.912

very beautifully confirms also the Bohr-Stoner scheme of the building-on of the various electrons of different elements in this group, and fits exactly

as would be expected into the new theory brought forward by Hund[4] on the elements of the first long period.

The position of the lines in Sc_{III} were predicted at $\nu = 62{,}000$ by an almost linear extrapolation of the 3^2D_2 term, the $4^2P_{1,2}$ terms being known from previous work.[1] Ireton,[5] having photographed the spark spectrum of scandium in this region using a vacuum spectrograph, gave us the connecting link between the data given by Fowler[3] for K_I and Ca_{II} and the data we already had on titanium and vanadium.[6]

TABLE 2

TITANIUM IV

INT.	λ	ν	Δν	TERMS
6	2103.79	47533.3		$4^2P_2-4^2S_1$
			817.5	
6	2068.22	48350.8		$4^2P_1-4^2S_1$
5	781.79	127911.6		$3^2D_2-4^2P_1$
8	779.09	128354.9	813.3	$3^2D_3-4^2P_2$
1	776.85	128724.9	370.0	$3^2D_2-4^2P_2$

TERM	VALUE	$\sqrt{\nu/R}$
4^2S_1	281840	1.602
4^2P_1	234307	1.461
4^2P_2	233489	1.442
3^2D_2	362219	1.817
3^2D_3	361844	1.816

TABLE 3

VANADIUM V

INT.	λ	ν	Δν	TERMS
7	1715.82	58281.1		$4^2P_2-4^2S_1$
			1265.4	
7	1679.36	59546.5		$4^2P_1-4^2S_1$
4	484.47	206411.1		$3^2D_3-4^2P_1$
5	482.99	207043.6	1277.5	$3^2D_3-4^2P_2$
2	481.49	207688.6	645.0	$3^2D_2-4^2P_2$

TERM	VALUES	$\sqrt{\nu/R}$
4^2S_1	408370	1.929
4^2P_1	350089	1.786
4^2P_2	348824	1.783
3^2D_2	556512	2.252
3^2D_3	555867	2.250

In Sc_{III} the jump from $4^2P_{1,2}$ to $3^2D_{2,3}$ results in an inverted diffuse doublet with one satellite, having an outside separation the same as the principal lines 4^2S_1–$4^2P_{1,2}$. These three lines were easily identified by their relative intensities and their known separation. These terms plotted

on the Moseley diagram show that $\sqrt{\nu/R}$ progresses almost linearly with atomic number as was previously expected.

Having worked out the spectrum of titanium and vanadium below wave-length 1500 A.U., the similar jumps for Ti_{IV} and V_V were readily identified. The lines thus found are given in tables 1, 2 and 3 with their frequencies, and the term values computed from the $4^2P_{1,2}$ and 4^2S_1 terms given in a previous report.[1]

With these short wave-lengths the frequencies of lines rise so rapidly with decreasing wave-lengths that fairly high dispersion and good resolution is necessary for accurate measurements, so that the observed $\Delta\nu s$, here given, are well within the experimental error of measurement. It is worthy of note that even in the case of V_V, where four electrons have been completely removed from the atom, the lines arising from the last electron are relatively strong except for the line of shortest wave-length which is fairly weak.

Recently, Bowen and Millikan,[7] working on the second long period, have identified the corresponding levels in Yt_{III} and Zr_{IV} which show exactly the same behavior as in the first long period, namely, that the "d" electrons are the tightest bound causing the $4^2D_{2,3}$ term values to rise above the others in Yt_{III}.

In our report[8] on the second and third long periods only the $^2P_{1,2}$ and $^2S_1$ levels had been located. However, we have now identified the $5^2D_{2,3}$–$6^2P_{1,2}$ lines in La_{III} which show again that the $5^2D_{2,3}$ term values rise above the $6^2P_{1,2}$ and 6^2S_1 term values and, therefore, that the "d" electrons are more tightly bound than either the "p" or the "s" electrons.

As yet we have not been able to locate the ^2F levels of the first long period because of their small wave-length separations even though their positions can be predicted to within two or three Ångströms of their correct value. We hope by photographing them in higher orders with a new vacuum spectrograph now being constructed to identify and establish the ^2F terms.

[1] Gibbs, R. C. and White, H. E., *Proc. Nat. Acad. Sci.*, **12**, p. 448, 1926.

[2] Russell, H. N., and Saunders, F. A., *Astrophys. Journ.*, **61**, p. 66, 1925.

[3] Fowler, A., *A Series in Line Spectra*, Fleetway Press, 1922.

[4] Hund, F., *Zs. f. Phys.*, **33**, p. 345, 1925.

[5] Ireton, H. J. C., *Trans. Roy. Soc. of Canada*, **18**, p. 103, 1924.

[6] From plates photographed by us at the Norman Bridge Laboratory in Pasadena.

[7] Bowen, I. S., and Millikan, R. A., Data not yet published, but mentioned here with their kind permission. Will appear in *Physic. Rev.*, Nov., 1926.

[8] Gibbs, R. C., and White, H. E., *Proc. Nat. Acad. Sci.*, **12**, p. 551, 1926.

THE ROTATIONAL ENERGY OF THE POLYATOMIC MOLECULE AS AN EXPLICIT FUNCTION OF THE QUANTUM NUMBERS

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In quantizing the rotational motion a molecule may be regarded for a first approximation as a rigid body[1] with three principal moments of inertia. The symmetrical case when two of the principal moments of inertia are equal presents no mathematical difficulties and yields a very simple formula for the rotational energy as a function of the quantum numbers. This was first derived by Schwarzschild.[2] The general asymmetrical case when no two of the principal moments of inertia are equal is considerably more difficult, and has been treated by Epstein,[3] Reiche[4] and Kramers.[5] The system being degenerate, there are several methods of quantization. The present discussion is based on the method used by Epstein and Kramers, which has the advantage of simplicity. Of the two phase integrals involved one is easily evaluated, but the other one is an elliptic integral, which contains the energy, W, as a parameter. In this paper the latter integral is evaluated for all values of the parameters that have physical significance, and the energy, W, is obtained as an explicit function of the quantum numbers.[6]

A brief sketch of the dynamics will first be given. The notation will be that of Born's "Atommechanik" (p. 126 ff.), from which a number of formulas will be taken. This discussion will be limited to the case where the total electronic angular momentum is zero. The molecule, regarded as a rigid body, has three principal moments of inertia, A_x, A_y, A_z,† the axes of which are taken as the x, y, z axes, respectively. These axes are fixed in the body. $(\bar{x}, \bar{y}, \bar{z})$ represents another coördinate system fixed in space, the origin being the center of gravity of the body. The $\bar{z}$-axis will be taken to coincide with the vector of total angular momentum, D. The relation of these two coördinate systems is defined by Euler's angles: θ, the angle between the z and $\bar{z}$ axes; φ, the angle between the x-axis and the line of nodes (which is the intersection of the xy plane with the $\bar{x}\bar{y}$ plane); ψ, the angle between the $\bar{x}$-axis and the line of nodes. The conjugate momenta are p_θ, p_ϕ and p_ψ, respectively.

The components of the total angular momentum, D_x, D_y, D_z, satisfy the equation

$$D^2 = D_x^2 + D_y^2 + D_z^2. \qquad (1)$$

The total energy, W, is here kinetic and is given by the expression

† Throughout this paper it will be assumed that no two of these quantities are equal.

$$W = \frac{1}{2}\left[\frac{D_x^2}{A_x} + \frac{D_y^2}{A_y} + \frac{D_z^2}{A_z}\right]. \tag{2}$$

In any given case W and D are constants. The sphere (1) and the ellipsoid (2) intersect in a curve, which we shall call C. This curve is the path traversed by the extremity of the vector of total angular momentum, D, referred to the axes x, y, z. The angles θ and φ may be regarded as, respectively, the co-latitude and longitude of the vector D referred to the z-axis as polar axis.

The two quantum conditions are

$$D = \frac{mh}{2\pi}. \tag{3}$$

$$D \oint \cos\theta \, d\varphi = nh \tag{4}$$

where m and n are integers. Equation (4) may be expressed in the form

$$\oint \sqrt{\frac{2W - D^2\left(\frac{\sin^2\varphi}{A_x} + \frac{\cos^2\varphi}{A_y}\right)}{\frac{1}{A_z} - \left(\frac{\sin^2\varphi}{A_x} + \frac{\cos^2\varphi}{A_y}\right)}}\, d\varphi = nh. \tag{5}$$

The integral on the left is the elliptic integral referred to above.

Let

$$\left.\begin{aligned} w &= 2W - \frac{D^2}{2}\left(\frac{1}{A_x} + \frac{1}{A_y}\right) \\ c &= \frac{1}{A_z} - \frac{1}{2}\left(\frac{1}{A_x} + \frac{1}{A_y}\right) \\ a &= \frac{1}{2}\left(\frac{1}{A_x} - \frac{1}{A_y}\right) \\ u &= \frac{w}{c} \\ b &= \frac{a}{c} \\ k &= \frac{D^2}{u} \end{aligned}\right\} \tag{6}$$

Then equation (5) may be reduced to the form

$$\sqrt{u} \oint \frac{\sqrt{1 + kb\cos 2\varphi}}{\sqrt{1 + b\cos 2\varphi}}\, d\phi = nh, \tag{7}$$

k and b are dimensionless quantities.

It will now be shown that in every real case the x, y, z axes may be so chosen that $|kb| \leqslant 1$ and $|b| < 1$, for the curve C gives θ as a function of φ. Since the sphere (1) and the ellipsoid (2) are concentric, C is a symmetrical closed curve, having as axis of symmetry either the longest or the shortest of the principal axes of the ellipsoid (2). Using this axis of symmetry as the z-axis, $\theta \leqslant \pi/2$ and ϕ is a monotone increasing function of the time. It can easily be shown from equations (6) that with this choice of axes $|b| < 1$. That $|kb| \leqslant 1$ follows from the fact that the integrand in equations (4) and (7) must be real for all values of ϕ that actually occur in the motion. With this choice of axes ϕ takes on all values, so that $|kb|$ must be less than or equal to 1.

It can easily be proved that when $|kb| = 1$, if $A_x < A_y < A_z$ or $A_x > A_y > A_z$,

$$W = \frac{m^2h^2}{8\pi^2A_y}. \tag{8}$$

The quantum number n does not appear in this formula. The phase integral (5) or (7) in this case reduces to

$$m\frac{2}{\pi}\sin^{-1}\sqrt{\frac{|2b|}{1+|b|}} = m\frac{2}{\pi}\sin^{-1}\sqrt{\frac{\frac{1}{A_y}-\frac{1}{A_x}}{\frac{1}{A_z}-\frac{1}{A_x}}} = n, \tag{9}$$

so that a stationary state corresponding to (8) exists only if equation (9) is satisfied for some integer n.

The only case that remains is the one where

$$|kb| < 1$$
$$|b| < 1.$$

In this case both the numerator and denominator of the integrand in equation (7) may be expanded by the binomial theorem and the two series multiplied together. The resulting series is absolutely and uniformly convergent, and may, therefore, be integrated term by term. The limits of integration are 0 and 2π, and the result obtained is

$$\frac{nh}{2\pi} = \sqrt{u}\left\{1 - \frac{1}{16}(k^2+2k-3)b^2 - \frac{3}{1024}(5k^4+4k^3+6k^2+20k-35)b^4+\ldots.\right\}. \tag{10}$$

Let

$$N = \frac{nh}{2\pi}. \tag{11}$$

Then, squaring equation (10), and using equations (6),

$$N^2 = u\left\{1 - \frac{1}{8}\left(\frac{D^4}{u^2} + 2\frac{D^2}{u} - 3\right)b^2 - \frac{1}{512}\left(13\frac{D^8}{u^4} + 4\frac{D^6}{u^3} + 22\frac{D^4}{u^2} + 84\frac{D^2}{u} - 123\right)b^4 + \ldots.\right\}. \quad (12)$$

Referring to equations (6) it will be seen that u contains the energy, W. Equation (12) must, therefore, be solved for u. As will be proved below, u, regarded as a function of b, is analytic in the neighborhood of $b = 0$, and may, therefore, be expanded in a Taylor's series about that point.

$$u = u_0 + \left(\frac{du}{db}\right)_0 b + \frac{1}{2!}\left(\frac{d^2u}{db^2}\right)_0 b^2 + \ldots. \quad (13)$$

$$u_0 = N^2. \quad (14)$$

By removing the brackets in equation (12), differentiating r times with respect to b, and setting $b = 0$, one obtains

$$0 = \left(\frac{d^r u}{db^r}\right)_0 + \left[\text{a finite number of terms of the form } \frac{D^{2\mu}}{u_0^\lambda}\left(\frac{d^{k_1}u}{db^{k_1}}\right)_0^{p_1} \ldots. \left(\frac{d^{k_\alpha}u}{db^{k_\alpha}}\right)_0^{p_\alpha}\right], \quad (15)$$

where $$k_1p_1 + \ldots. k_\alpha p_\alpha < r, \quad (16)$$

and $$\lambda \geqslant -1.$$

With the present choice of axes the case $n = 0$ cannot occur in the asymmetrical molecule, unless $D = m = 0$. Hence from equation (14) u_0 is always different from zero. It follows from equation (15) that all the derivatives exist and u is analytic at the point $b = 0$. The derivatives of odd order vanish at $b = 0$—a fact easily demonstrated by the use of equation (15). Using the same equation, the derivatives of even order may be obtained one by one, beginning with the 2nd, then getting the 4th, etc.

It is evident from this process that

$$\left(\frac{d^{2r}u}{db^{2r}}\right) = (2r)!\, f_r(D^2, u_0) = (2r)!\, f_r(D^2, N^2), \quad (17)$$

where f_r is a polynomial of the rth degree in D^2.

Then

$$u = \sum_{r=0}^{\infty} f_r(D^2, N^2)b^{2r}. \quad (18)$$

Since u is of the same dimensions as D^2 and N^2, and b dimensionless, f_r must be of the form

$$f_r = \sum_{s \leqslant r} a_s D^{2s} N^{2-2s}, \tag{19}$$

where the a_ss are numerical coefficients. But D^2 in equation (12) occurs only with exponents $\alpha \geqslant 0$. It is easily seen from equation (15) that the same is true in $f_r(D^2, N^2)$. It follows that in equation (19) $s \geqslant 0$.

Using equations (3) and (11) in equation (18),

$$u = \frac{h^2}{4\pi^2} \sum_{r=0}^{\infty} f_r(m^2, n^2) b^{2r}, \tag{20}$$

where

$$f_r(m^2, n^2) = \sum_{s=0}^{r} a_s \frac{m^{2s}}{n^{2s-2}}. \tag{21}$$

Then from equations (6)

$$W = \frac{h^2}{8\pi^2} \left\{ \frac{m^2}{2} \left(\frac{1}{A_x} + \frac{1}{A_y} \right) + n^2 c + \sum_{r=1}^{\infty} f_r(m^2, n^2) \frac{a^{2r}}{c^{2r-1}} \right\}. \tag{22}$$

By the process above as many of the f_rs may be determined as is desired. The first two are

$$\left.\begin{aligned} f_1 &= \frac{1}{8} \left(\frac{m^4}{n^2} + 2m^2 - 3n^2 \right) \\ f_2 &= \frac{1}{512} \left(5 \frac{m^8}{n^6} - 12 \frac{m^6}{n^4} + 22 \frac{m^4}{n^2} + 36m^2 - 51n^2 \right). \end{aligned}\right\} \tag{23}$$

In equation (21) $\sum_{s=0}^{r} a_s = 0$, a fact illustrated by equations (23).

For any given value of D or m there are a number of values of W. If A_x, A_y, A_z are the principal moments of inertia in increasing order of magnitude, then the least possible value of W is given by $n = m$ in equation (22), which then reduces to

$$W = \frac{m^2 h^2}{8\pi^2 A_z}. \tag{24}$$

Higher values of W in increasing order of magnitude are obtained by using in equation (22) the values of $n < m$ in decreasing order of magnitude until the least possible value of n, which we shall designate $n_0^{(m)}$, has been reached. Defining (cf. equation (9)) $n^{(m)}$ as

$$n^{(m)} = m\frac{2}{\pi}\sin^{-1}\sqrt{\frac{|2a|}{|a|+|c|}} = m\frac{2}{\pi}\sin^{-1}\sqrt{\frac{\frac{1}{A_y}-\frac{1}{A_x}}{\frac{1}{A_z}-\frac{1}{A_x}}}, \quad (25)$$

$$n_0^{(m)} = [n^{(m)}], \quad (26)$$

where the symbol $[t]$ means the least integer greater than or equal to t. If in equation (25) $n^{(m)}$ is integral, $|kb| = 1$ for $n = n^{(m)}$ and equation (8) must then be used instead of equation (22).

For the remaining higher states the rôle of x and z axes must be interchanged, and the values of n beginning with the least possible, $\bar{n}_0^{(m)}$, inserted in the formula obtained from equation (22) by this interchange.

$$\bar{n}_0^{(m)} = [\bar{n}^{(m)}], \quad (27)$$

where

$$\bar{n}^{(m)} = m\frac{2}{\pi}\sin^{-1}\sqrt{\frac{\frac{1}{A_y}-\frac{1}{A_z}}{\frac{1}{A_x}-\frac{1}{A_z}}}. \quad (28)$$

$$n^{(m)} + \bar{n}^{(m)} = m. \quad (29)$$

Finally when $n = m$ again

$$W = \frac{m^2h^2}{8\pi^2A_x}, \quad (30)$$

which is the greatest possible value of W. The number of values of W for any given value of m is $(m + 1)$.

The selection principles for m and n are:

$$\Delta m = 0, \pm 1.$$

If the electric moment is in the direction of the z-axis, then

$$\Delta n = 0, \pm 2, \pm 4, \pm 6, \text{ etc.}$$

If the electric moment lies in the xy plane,

$$\Delta n = \pm 1, \pm 3, \pm 5, \pm 7, \text{ etc.}$$

In a paper to be published elsewhere further details of the preceding mathematical theory will be given, the selection principles deduced, and formulas developed for the relative intensity of the various bands due to different transitions in n.

The author wishes to express his thanks to Prof. E. C. Kemble, who suggested this investigation.

* NATIONAL RESEARCH FELLOW.

[1] M. Born and W. Heisenberg, *Ann. d. Physik*, **74**, 1, 1924.

[2] K. Schwarzschild, *Ber. d. Berliner Akad.*, **1916**, p. 548.

[3] P. S. Epstein, *Verh. d. Deutsch Phys. Ges.*, **18**, 398, 1916; *Phys. Zeits.*, **20**, 289, 1919.

[4] F. Reiche, *Phys. Zeits.*, **19**, 394, 1918.

[5] H. A. Kramers, *Zeits. f. Physik*, **13**, 343, 1923.

[6] When the work here presented was done, there was no published work dealing with this problem, except a first approximation by Reiche (loc. cit.) for the case when the asymmetry is small. Quite recently, however, F. Lütgemeier (*Zeits. f. Physik*, **38**, 251, 1926) has published a solution of it. The present mode of treatment and the resulting formula for W are different from his. The formula for W here derived contains only even powers of the quantum numbers, whereas his contain both even and odd powers. While his two formulas are doubtless equivalent to the one developed here, the differences referred to seemed to justify the publication of our results.

THE DIELECTRIC CONSTANT OF SYMMETRICAL POLYATOMIC DIPOLE-GASES ON THE NEW QUANTUM MECHANICS

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In a recent note in these PROCEEDINGS[1] the problem of the dielectric constant of diatomic dipole-gases has been discussed from the viewpoint of the new quantum mechanics. In the meantime a paper by Dennison[2] has appeared, in which frequencies and amplitudes are computed for the more general system of a rigid polyatomic molecule possessing an axis of symmetry and hence two equal moments of inertia. It is, therefore, possible now to deal also with the question of the dielectric constant of such a molecule with a permanent electric moment μ. The problem is simplified if it be remembered that due to the symmetry the electric moment will lie along the axis of the molecule. An example is probably NH_3 with the three H-nuclei at the corners of an equilateral triangle forming the base of a symmetrical pyramid at the apex of which the N-nucleus is located.

Following Dennison's notation we shall call the moment of inertia about the axis of figure C, the other one A. In his paper, to which we must refer concerning all details, it is shown that the system under consideration is characterized by three quantum numbers j, n, m (there called m, n, σ), if a uniform precession be superimposed on the body to remove the degeneracy of spacial orientation. The first of these determines the total moment of momentum, the second the moment of momentum about the axis of symmetry, while the last represents in units $h/2\pi$ the moment of momen-

tum along the axis of precession z. If the energy differences due to the precession be neglected, the energy of the state j, n, m is given by

$$W^0_{j,n} = \left[\frac{1}{A}j(j+1) + \left(\frac{1}{C} - \frac{1}{A}\right)n^2 + \text{const.}\right]\frac{h^2}{8\pi^2}. \qquad (1)$$

In order that this expression for $C = 0$, $n = 0$ may become identical in its dependence on j with that for the simple rotator in space, j must have integral values. For a given value of j, n and m may have all integral values obeying the restrictions

$$|n| \leqq j, \qquad |m| \leqq j.$$

Since the electrical moment lies along the axis of symmetry, the transitions involving the frequency of rotation about this axis have zero amplitudes; i.e., radiative transitions in which n changes do not occur. With this in mind it is seen from Dennison's results that in the vector matrix representing the electric moment of the moving system, the only terms different from zero are those corresponding to transitions $(j, n, m; j, n, m)$, $(j, n, m; j-1, n, m)$, $(j, n, m; j-1, n, m \pm 1)$, $(j, n, m; j, n, m-1)$. Of these again only the first two give rise to terms in the z-component $\mathbf{P}^0_z$ of the electrical moment. From (1) and the frequency condition we find their frequency to be

$$\nu(j, n; j-1, n) = -\nu(j-1, n; j, n) = \frac{h}{4\pi^2 A}j \qquad (2)$$

if again the precessional frequency be neglected. Furthermore the corresponding amplitudes are

$$\left.\begin{aligned} P^0_z(j, n, m; j, n, m) &= \frac{nm}{j(j+1)}\mu, \\ P^0_z(j, n, m; j-1, n, m) &= \sqrt{\frac{(j^2-n^2)(j^2-m^2)}{j^2(2j-1)(2j+1)}}\,\mu. \end{aligned}\right\} \qquad (3)$$

The procedure to find the dielectric constant due to the permanent moments of the molecules is entirely analogous to that used in the case of diatomic molecules. From the perturbation theory[3] it follows that if a constant electric field E is applied parallel to z, the matrix $\mathbf{P}_z$ is now given in first approximation by

$$\mathbf{P}_z = \mathbf{P}^0_z + \mathbf{P}^1_z$$

and hence the terms independent of the time $P_z(j, n, m; j, n, m)$ by

$$P_z(j, n, m; j, n, m) = P^0_z(j, n, m; j, n, m) + P^1_z(j, n, m; j, n, m), \qquad (4)$$

where

$$P_z^1(j,n,m;\ j,n,m) = -\frac{2E}{h}\sum_{j',n',m}{}' \frac{P_z^0(j,n,m;\ j',n',m')\ P_z^0(j',n',m';\ j,n,m)}{\nu(j,n,m;\ j',n',m')}, \quad (5)$$

the accent at the summation sign indicating that j', n', m' must not have the values j, n, m simultaneously. Similarly the energy values of the stationary states of the perturbed system are in first approximation

$$W_{j,n,m} = W_{j,n}^0 + W_{j,n,m}^1, \quad (6)$$

where

$$W_{j,n,m}^1 = -E\,P_z^0(j,n,m;\ j,n,m). \quad (7)$$

By introducing (2) and (3) in (4) and (5) we get

$$P_z(j,n,m;\ j,n,m) = \frac{nm}{j(j+1)}\mu - \frac{8\pi^2 AE\mu^2}{h^2}\left\{\frac{(j^2-n^2)(j^2-m^2)}{j^3(2j-1)(2j+1)} - \frac{[(j+1)^2-n^2][(j+1)^2-m^2]}{(j+1)^3(2j+1)(2j+3)}\right\}.$$

The probability of an atom being in the state j, n, m is

$$\frac{e^{-W_{j,n,m}/kT}}{\sum_{j,n,m} e^{-W_{j,n,m}/kT}},$$

and the dielectric constant ϵ due to the permanent moments of the molecules is then given by

$$\frac{3}{4\pi}\frac{\epsilon-1}{\epsilon+2} = \frac{N\sum_{j,n,m} P_z(j,n,m;\ j,n,m)e^{-W_{j,n,m}/kT}}{E\sum_{j,n,m} e^{-W_{j,n,m}/kT}}, \quad (8)$$

where N is the number of molecules per cm.^{3} The sum in the numerator we wish to compute to terms linear in E. We may hence write for it with the aid of (4) and (6)

$$\sum_{j,n,m}[P_z^0(j,n,m;\ j,n,m) + P_z^1(j,n,m;\ j,n,m)]e^{-(W_{j,n}^0 + W_{j,n,m}^1)/kT}$$

$$= \sum_{j,n,m}[P_z^0(j,n,m;\ j,n,m) + P_z^1(j,n,m;\ j,n,m)]\left(1 - \frac{W_{j,n,m}^1}{kT}\right)e^{-W_{j,n}^0/kT}$$

$$= \sum_{j,n,m}[P_z^0(j,n,m;\ j,n,m) - P_z^0(j,n,m;\ j,n,m)\frac{W_{j,n,m}^1}{kT} + P_z^1(j,n,m;\ j,n,m)]e^{-W_{j,n}^0/kT}.$$

Substituting for P_z^0, P_z^1 and W^1 their values from (3), (5) and (7), and summing over m from $-j$ to j, we get zero from the first term while the other two give

$$\frac{E\mu^2}{3kT}\sum_{j,n}\left[\frac{n^2(2j+1)}{j(j+1)}+\frac{1}{\alpha}\frac{n^2(2j+1)}{j^2(j+1)^2}\right]e^{-\alpha j(j+1)-\beta n^2} \qquad (9)$$

if we introduce the abbreviations

$$\alpha=\frac{h^2}{8\pi^2kT}\frac{1}{A},\qquad \beta=\frac{h^2}{8\pi^2kT}\left(\frac{1}{C}-\frac{1}{A}\right).$$

In the sum in the denominator of (8) we may put $W^0_{j,n}$ instead of $W_{j,n,m}$ and get hence

$$\sum_{j,n}(2j+1)e^{-\alpha j(j+1)-\beta n^2}. \qquad (10)$$

The two sums (9) and (10) we may approximate by integrals if α and β are small compared to unity, i.e., if T is sufficiently large. They become, respectively,

$$\frac{2E\mu^2}{3kT}\int_0^\infty dn.n^2e^{-\beta n^2}\int_n^\infty dj.(2j+1)\left[\frac{1}{j(j+1)}+\frac{1}{\alpha j^2(j+1)^2}\right]e^{-\alpha j(j+1)} \qquad (11).$$

$$2\int_0^\infty dn.e^{-\beta n^2}\int_n^\infty dj.(2j+1)e^{-\alpha j(j+1)}. \qquad (12)$$

The integrations with respect to j can be performed if we introduce as new variable $\alpha j(j+1)=x$. Then we have in the first integral from integration by parts

$$\int_n^\infty dj.(2j+1)\left[\frac{1}{j(j+1)}+\frac{1}{\alpha j^2(j+1)^2}\right]e^{-\alpha j(j+1)}$$

$$=\int_{\alpha n(n+1)}^\infty dx.\left(\frac{1}{x}+\frac{1}{x^2}\right)e^{-x}=-\frac{1}{x}e^{-x}\Big]_{\alpha n(n+1)}^\infty=\frac{1}{\alpha n(n+1)}e^{-\alpha n(n+1)},$$

in the second

$$\int_n^\infty dj.(2j+1)e^{-\alpha j(j+1)}=\frac{1}{\alpha}\int_{\alpha n(n+1)}^\infty dx.e^{-x}=\frac{1}{\alpha}e^{-\alpha n(n+1)}.$$

The first expression (11) thus becomes

$$\frac{2E\mu^2}{3kT\alpha}\int_0^\infty dn.\frac{n}{n+1}e^{-\alpha n(n+1)-\beta n^2},$$

the expression (12)

$$\frac{2}{\alpha}\int_0^\infty dn.e^{-\alpha n(n+1)-\beta n^2}.$$

In the former $n/(n+1)$ may be replaced without error by unity in the approximation we are interested in. Substituting the values so obtained for the sums in (8) gives

$$\frac{3}{4\pi}\frac{\epsilon-1}{\epsilon+2}=\frac{N\mu^2}{3kT}.$$

We see then that here, just as in the case of diatomic molecules, the permanent moment of the molecules and the part of the dielectric constant due to it are related by Debye's equation derived on the classical theory provided the temperature is sufficiently high.

As regards the behavior of the bands emitted by dipole-molecules of the kind considered in an electric field, there should be a linear Stark effect for all lines arising from transitions between states for which n is different from zero, as may be seen directly from (7) and (3). The deflection of the molecules in an electric Stern-Gerlach experiment of the kind described in the previous paper[1] can be easily predicted, too, by introducing the value of P determined by equations (4), (3) and (5) into the formula for the deflection.

[1] R. de L. Kronig, *Proc. Nat. Acad. Sci.*, **12**, 488, 1926. See also L. Mensing and W. Pauli, *Phys. Zs.*, **27**, 509, 1926; J. H. Van Vleck, *Nature*, **118**, 226, 1926.

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[3] M. Born, W. Heisenberg and P. Jordan, *Zs. f. Phys.*, **35**, 557, 1926.

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*TROPISMS OF MAMMALS**

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Conceptions of the mechanism of adjustor function in the central nervous system have chiefly been based upon results of "learning" tests, as in a maze, and upon the method of conditioned reflexes. These modes of experimentation are cumbersome, and do not give direct approach to the understanding of variability in conduct and the basis of moment-to-moment adjustment in behavior. On the other hand, interpretation in terms of reflexes has obvious limitations.

We desired to see if certain conceptions of the analysis of conduct, as derived from study of lower animals, might not after all be applicable in this connection, and to obtain if possible another route of approach; one which would permit the expression of behavior in quantitative terms and make possible the dynamical formulation of conduct in situations

determined by the conflict of alternative types of response. If an organism be exposed to two sources of stimulation the resultant movements may be controlled by the summation or balance of the respective excitations; or, with other suitable experimental arrangements, the organism's movement may be decided by one of the two excitations to the complete exclusion of the effect of the other.[1] Such a "decision" may be taken to represent an elementary act of central nervous adjustment, and, given rational units in which to express the effects of the two sources of excitation when acting singly, its nature may be investigated by measuring its quantitative modification under controlled conditions. Thus with the negatively phototropic larva of the beetle *Tenebrio*[2] the arrangement can be made such that the intensity of light may be measured which is just sufficient to suppress the animal's positive stereotropism.

The development of such a method for investigation of central nervous functions in higher animals, mammals for example, requires the identification in them of mathematically predictable modes of behavior. Although it has been loosely suggested[3] that "ideas" and "memory images" may serve as sources of tropistic guidance, it is fair to assert that little more than figurative progress has been made toward the study of tropistic conduct in mammals. Our own experiments suggest that absence of information on this topic is due, partly, to the fact that the necessary and very simple experiments seem not to have been made hitherto.

It proves possible to recognize in the behavior of young mammals very definite modes of tropistic behavior. We have employed nestling rats and mice. Careful attention must be given to the age and to the genetic uniformity of the experimental material. In fact, we believe that within certain limits the resulting methods may be used for the exact comparison of genetically diverse stocks.

It turns out that in several respects it becomes feasible, on the basis of these experiments, to refine and to extend the general account of phototropism and of geotropism as modes of behavior. We confess a certain satisfaction in being able to reverse to this extent the traditional argument that behavior-analysis is to be attained first with "lower" animals.

The vector character of stereotropism in young rats and mice is readily demonstrated. When creeping in contact with the outer surface of a box or a solid block the rat turns the corner and continues to follow the vertical surface. If two blocks make contact, one with either side of the rat, there is no swerving to either side on emergence from the zone of contacts.[4] The result is, therefore, identical with that in similar experiments upon arthropods.[5]

Phototropism and geotropism may be examined in a manner permitting exact statement of relations between the intensity of the acting force and the amount of the response; this is not possible with stereotropism. Young

rats and mice, before the eyelids have parted, move away from a source of horizontal light; after the eyes are opened, the situation is quite different, due to image formation. The negative orientation is precise. That it is a truly tropistic orientation is proved in two ways.[6] If one eye be removed, illumination from above results in the continuous execution of circus movements toward the blind side. The postures of the limbs, showing heightened extensor tone on the non-blind side, recalls in all essentials the limb positions of an insect such as *Ranatra*.[7] On the other hand, if exposed to two diametrically opposed sources of light the rat moves in a path at an angle to the line joining the two sources. This angle may be calculated[8] from the measured intensities of the two lights, by mean of the basic assumption of the phototropism doctrine. This assumption is that the animal ceases to orient when the illumination on its two sides is equal; under the conditions given, orientation is possible for a negatively heliotropic animal when its photoreceptive surfaces are non-parallel. The basic assumption leads to the equation

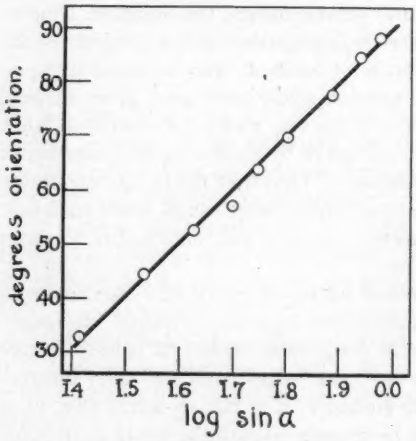

FIGURE 1

The amount of upward orientation of young rats, measured by the angle (θ) of the path with the horizontal, when creeping upon a wire grid inclined at angle α with the ground. The average orientation is directly proportional to the logarithm of the effective gravitational attraction. (One point departs from the course of the rest by as much as twice its P.E.)

$$\tan\theta = \frac{(I_1 + I_2)}{(I_1 - I_2)} \cot\frac{H}{2},$$

where I_1 and I_2 are the respective photic intensities, θ is the observed angle of inclination to the line connecting the two lights, and H is the effective angle between the photosensitive surfaces. In the rat, increased total illumination $(I_1 + I_2)$ causes increased photokinetic "random" movement of the head; hence it is found that H decreases very nearly in proportion to log $(I_1 \times I_2)$.[8] (The formula given can be tested, with very satisfactory agreement, for cases in other animals where H does not vary with the intensity of illumination.)

The geotropism of the young rat is equally precise.[9] Allowed to creep upon a fine-meshed wire grid, orientation is always upward. The amount of orientation depends upon the angle of inclination of the creeping-surface. The path of progression is a straight line, more and more nearly at 90° orientation as the inclination of the creeping-plane is increased. The minimum inclination giving orientation is 10°+; at 70°+ inclination the angle of orientation on the creeping-plane is 90°. Between these limits the angle of orientation (θ) is directly proportional to the log (sin α), where α is the angle of inclination (Fig. 1). This proof of orientation as determined by the active component of gravity is checked by the fact that the coefficient of variation of the measurements at a given inclination decreases directly as log (sin α). Thus not only the amount, but also the precision of orientation, is determined by the effective gravitational attraction.

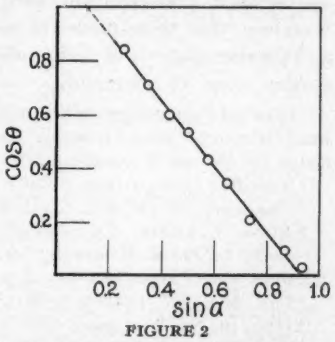

FIGURE 2

The amount of upward orientation (θ) of young rats upon a plane inclined at angle α is such that the ratio cos θ/sin α is sensibly constant.

The interpretation of this result is, simply, that a differential pull of the animal's weight upon the (leg?) muscles is responded to if it exceeds a certain threshold. This amounts to assuming that the difference between the pull of the animal's weight upon the legs of the opposite sides during creeping calls forth respectively pulling and pushing movements of the "upper" and the "lower" limbs, so that the rat turns until the distribution of the pull of the weight upon the opposite limbs is such that the difference between the two effects is a constant fraction of the total. This is a fair assumption because the amount of orientation is proportional to log (sin α); it leads to the expectation that the ratio $-\frac{\cos\theta,}{\sin\alpha}$ where θ is the angle of upward orientation, should be very nearly constant. Figure 2 shows that this expectation is thoroughly satisfied.

The further proof of this interpretation is given by the fact that the amplitude of orientation (θ) is increased by attaching weights to the animal's tail, and in proportion to the logarithm of the added mass.

Detailed consideration of these points is given in another place. We are now concerned merely to record the nature of the findings as regards these tropisms of young rats. They are as definite and predictable as in the most favorable instances among invertebrates, and they may be brought into opposition with one another in a similar way. Thus it is a fairly simple

matter to determine the photic intensity required to drive a rat away from stereotropically maintained contact with a glass plate, or to measure precisely the resolution of conflicts between stereotropism and geotropism, or between phototropism and geotropism. We consider it possible, therefore, that these modes of response may be employed for fairly direct and precise analysis of certain elementary functions of the central nervous system even of mammals.

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[1] *Year Bk., Carnegie Instn. Wash.; Jour. Gen. Physiol.*, **10**, No. 1, 1926.
[2] *Jour. Gen. Physiol.*, **6**, p. 531; p. 647.
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[4] *Jour. Gen. Physiol.*, **10**, No. 1, p. 195, 1926.
[5] *Ibid.*, **5**, p. 597, 1923; **6**, p. 531, 1924.
[6] *Ibid.*, **10**, No. 2, *in press*, 1926.
[7] *Ibid.*, **7**, p. 217, 1924.
[8] *Ibid.*, **10**, No. 2, *in press*, 1926.
[9] *Ibid.*, **10**, No. 2, *in press*, 1926.